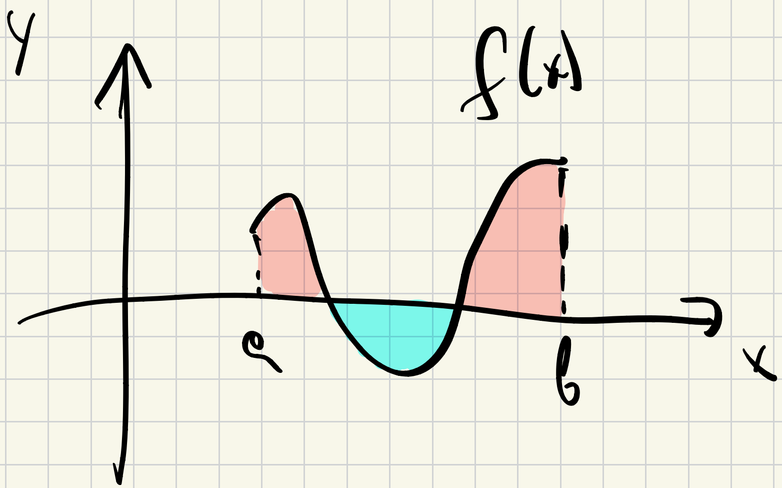
Analysis I Lecture 24

Last time:

Defined integral as volume under the graph of function:

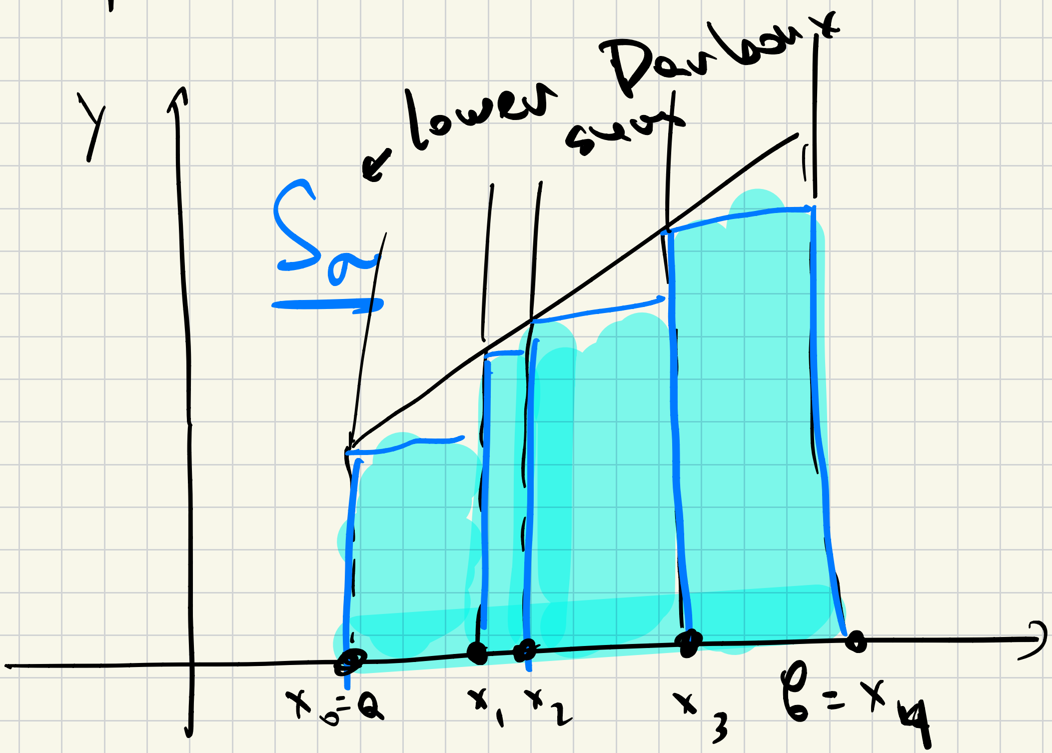


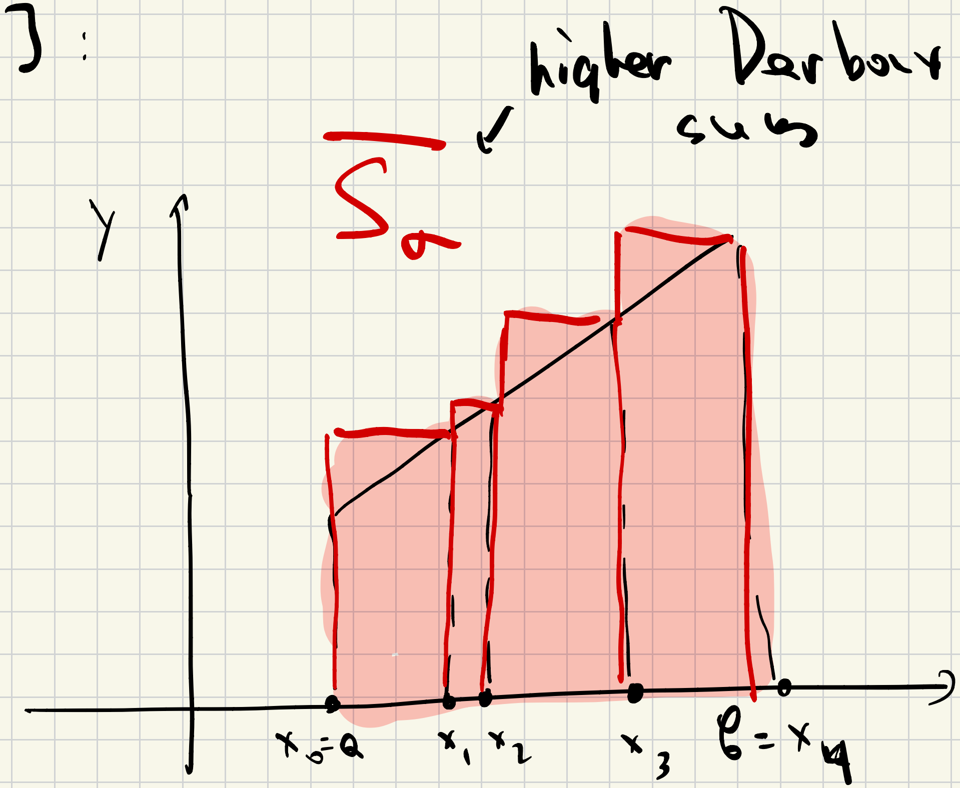
$$\int_a^b f(x)\,dx = \quad - \quad$$

Formal definition is via

Darboux sums corresponding to

partitions of $[a, b]$:





We say that f is integrable if:

$$\overline{S} = \inf \{ \overline{S}_\alpha \mid \alpha\text{- partition of } [a,b] \} =$$

upper Darboux integral

$$= \sup \{ \underline{S}_\alpha \mid \alpha\text{- partition of } [a,b] \}$$

Lower Darboux int. $\underline{S}$ "

Not all functions are integrable:

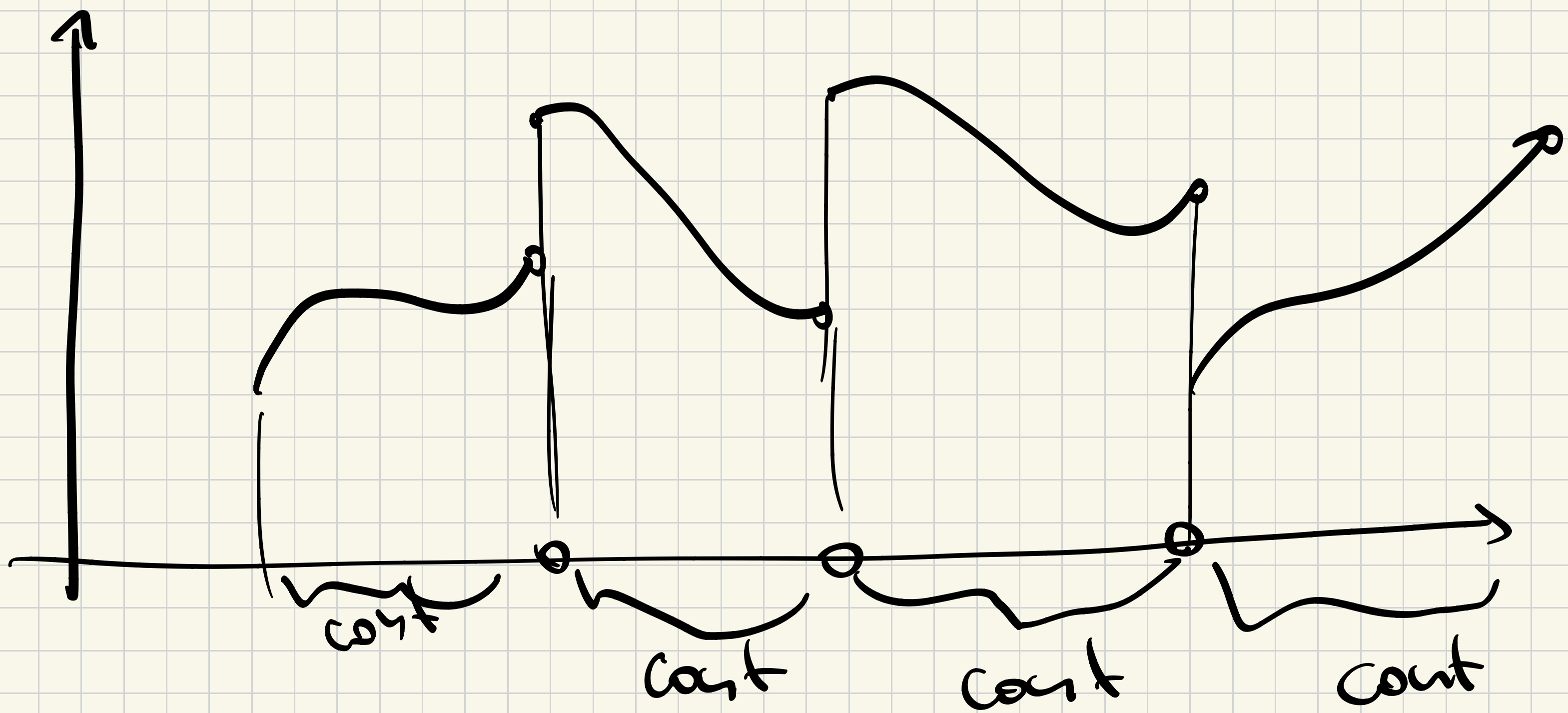
. $\chi_{\mathbb{Q}}$ is not integrable since

$$\underline{S_\sigma} = 0 \quad \text{and} \quad \overline{S_\sigma} = (b-a)$$

for any subdivision σ.

<u>Proposition</u> Let $f : [a,b] \to \mathbb{R}$ be continuous then it is integrable.

In fact, every piecewise continuous function is integrable:

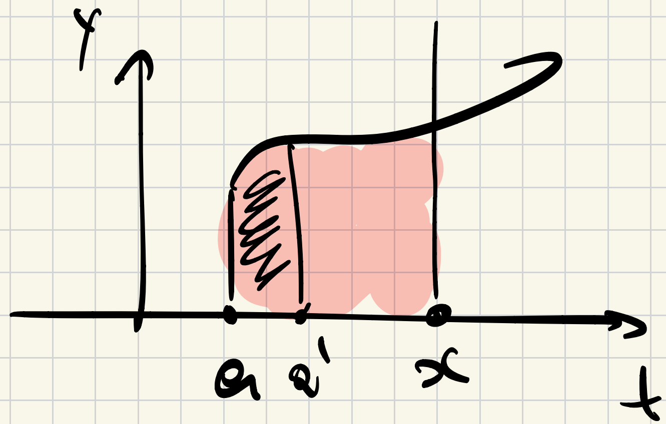


Fundamental theorem of calculus

Theorem 8.23 Let $f: [a,b] \to \mathbb{R}$ be continuous function. Define

$$F(x) = \int_a^x f(t)\,dt$$

x fixed



Then $F'(x) = f(x)$

In other words $F(x)$ is an anti-derivative of f.

Remark Antiderivative is not unique:

If $F'(x) = f(x)$ then

$$\forall c \in \mathbb{R} \quad (F+c)' = f(x)$$

So anti-derivative is only defined up to additive constant.

Theorem 8.24. Let $f : [a,b] \to \mathbb{R}$ be continuous function. and let $F(x)$ be its anti-derivative.

$F'(x) = f(x)$

Then
$$\boxed{\int_a^b f(x) = F(b) - F(a).}$$

Remark if we replace $F(x)$ by $F(x)+C$: RHS will not change! $(F(b)+C)-(F(a)+C)$
$= F(b) - F(a).$

Example

- $\displaystyle\int_a^b x^n \, dx = \frac{x^{n+1}}{n+1}\Bigg|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$

We use notation $f(x)\Big|_a^b = f(b) - f(a)$.

Anti-derivative for x^n is $\frac{x^{n+1}}{n+1}$

- $\displaystyle\int_a^b \sin(x) \, dx = (-\cos(x))\Big|_a^b = -\cos(b) - (-\cos(a)) = \cos(a) - \cos(b)$

since $(-\cos(x))' = \sin(x)$.

Today: How to compute integrals?

- Substitution
- By parts
- Partial Fractions. to integrate Rational functions.

Remark: Integration in finite terms

If we want to differentiate a function produced from a list of standard functions you always get a function in the same class.

by composing and arithmetic operations

polynomials, roots, trigonometric, log, exp ...

Integration is very different:

Not all anti-derivatives of elementary functions can be written as elementary functions!

Theorem 8.29 (Substitution or change of variables)

Let $f : [a,b] \to \mathbb{R}$ continuous and $\varphi : [\alpha,\beta] \longrightarrow [a,b]$ be a C^1-function then

$$\boxed{\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} \underline{f(\varphi(t)) \cdot \varphi'(t)}\,dt}$$

Proof:

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} \underline{f(\varphi(t)) \cdot \varphi'(t)}\,dt$$

Let $\boxed{G}$ be an anti-derivative of f.

Then LHS is equal to $G(x)\Big|_{\varphi(\alpha)}^{\varphi(\beta)} =$

$$= G(\varphi(\beta)) - G(\varphi(\alpha)).$$

So we need to show that

RHS is equal to $G(\varphi(\beta)) - G(\varphi(\alpha))$

$$= G(\varphi(t))\Big|_{\alpha}^{\beta}$$

But

$$\Big(G(\varphi(t))\Big)' = G'(\varphi(t)) \cdot \varphi'(t) =$$

$$= f(\varphi(t)) \cdot \varphi'(t)$$

So we get that

$G(\varphi(t))$ is an anti-derivative

of $f(\varphi(t)) \cdot \varphi'(t) \Rightarrow$

$$\Rightarrow \underbrace{\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\, dt}_{\text{RHS}} = G(\varphi(t))\Big|_\alpha^\beta$$

$\blacksquare$

Example

$$\int_0^1 \sqrt{1-x^2}\, dx =$$

substitute $x = \sin(t)$

$$\int_0^{\pi/2} \sqrt{1-\sin^2(t)} \cdot (\sin t)' \, dt =$$

$$= \int_0^{\pi/2} |\cos t| \cdot \cos(t) \, dt =$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t \, dt =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos(2t) + 1}{2} \, dt =$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} \frac{1}{2} \, dt}_{= \frac{x}{2} \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}} + \underbrace{\int_0^{\frac{\pi}{2}} \frac{\cos(2t)}{2} \, dt}_{=0} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos(2t)}{2}\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2t)\, dt =$$

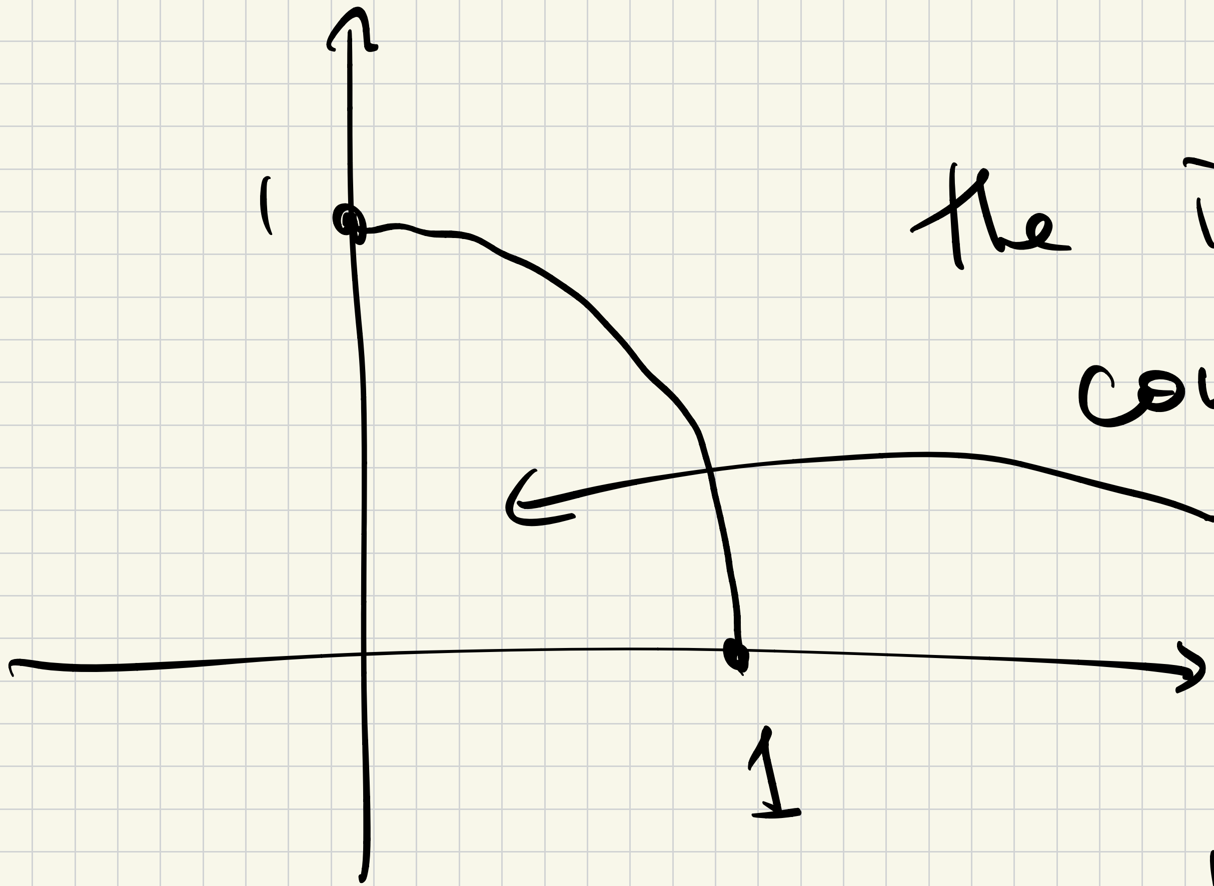
(we can guess antiderivative:

$$\left(\frac{\sin(2t)}{2}\right)' = \cancel{2}\cos(2t)$$
)

$$= \frac{1}{4}\cdot \sin(2t)\Big|_0^{\frac{\pi}{2}} = \frac{1}{4}\left(\sin(\pi) - \sin(0)\right)$$

$$= 0.$$

$$f(x) = \sqrt{1-x^2}$$



the integral computes Area of quater of a circle

$$\frac{1}{4}\pi \cdot r^2 = \frac{\pi}{4} \cdot r^2 = \frac{\pi}{4} \blacksquare$$

Example

$$e^1 = e \qquad f(x) = \sqrt{x} \qquad \varphi = e^x$$

$$\int_0^1 \sqrt{e^x} \cdot e^x \, dx = \int_{1 = e^0}^{e} \sqrt{u} \, du =$$

$u = e^x$

$$\left(\sqrt{e^x} \cdot e^x = \sqrt{\varphi} \cdot \varphi' \right)$$

$$\varphi \Rightarrow \varphi' = e^x$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_1^e = \frac{2}{3}\left(e^{\frac{3}{2}} - 1\right).$$

∎

Theorem 8.34 (Integration by parts)

Let $f, g : [a,b] \to \mathbb{R}$ be two C^1-functions

Then

$$\boxed{\int_a^b f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) \Big|_a^b - \int_a^b f'(x) \cdot g(x)\, dx}$$

Proof: Want to show that

$$\int_a^b f\cdot g'\,dx + \int_a^b f'\cdot g\,dx = fg\Big|_a^b$$

$$\int_a^b f\cdot g' + f' g\,dx = \int_a^b (f\cdot g)'\,dx$$

$\blacksquare$

Ways to use integration by parts

$$\int_a^b x \cdot e^x \, dx = \int_a^b \underbrace{x}_{f} \cdot \underbrace{(e^x)'}_{g} \, d =$$

$$e^x = (e^x)'$$

$$= x \cdot e^x \Big|_a^b - \int_a^b (x)' e^x \, dx =$$

$$= x e^x \Big|_a^b - \int_a^b e^x \, dx =$$

$$= \left(x e^x - e^x \right) \Big|_a^b = e^x (x-1) \Big|_a^b =$$

$$= e^b (b-1) - e^a (a-1).$$

More generally we can compute

$$\int_a^b \underbrace{x^n}_{f} \cdot \underbrace{e^x}_{g'} \, dx =$$

$g = e^x$

$$= x^n \cdot e^x \Big|_a^b - \int_a^b \underbrace{(x^n)'}_{= n \cdot x^{n-1}} e^x \, dx$$

So by applying enough time we are left with:

$$\underbrace{x^n \cdot e^x - n x^{n-1} \cdot e^x \cdots}_{\text{some explicit function}} \Big|_a^b \pm \int_a^b e^x \, dx$$

The same strategy also works

with $\int p(x) \cdot \begin{matrix} \sin(x) \\ \cos(x) \\ \sinh(x) \\ \cosh(x) \end{matrix} \, dx$

polynomial

Example

$$\underbrace{\int_a^b \underbrace{\sin(x)}_{f}\cdot\underbrace{e^x}_{g'}\,dx}_{\text{Int}} = \underbrace{\sin(x)}_{f}\cdot\underbrace{e^x}_{g}\Big|_a^b - \int_a^b \underbrace{+\cos(x)}_{f'}\,\underbrace{e^x}_{g}\,dx =$$

$$= \underbrace{\sin x\cdot e^x\Big|_a^b - \Big(\cos(x)\cdot e^x\Big|_a^b}_{\text{known}} - \underbrace{\int_a^b -\sin(x)\cdot e^x\,dx}_{\text{Int}}$$

So we get after 2 applications of IBP:

$$\int_a^b \sin(x) \cdot e^x \, dx = e^x (\sin(x) - \cos(x)) \Big|_a^b - \int_a^b \sin(x) \cdot e^x \, dx$$

$$\Rightarrow \boxed{\int_a^b \sin(x) e^x \, dx = \frac{e^x}{2} \cdot (\sin(x) - \cos(x)) \Big|_a^b}$$

Example

$$\int_a^b \log(x)\, dx = \int_a^b \underbrace{\log(x)}_{f} \cdot \underbrace{1}_{g'}\, dx =$$

$g' = 1 \Rightarrow g = x$

$$= \underbrace{\log(x)}_{f} \cdot \underbrace{x}_{g}\Big|_a^b - \int_a^b \underbrace{x}_{g} \cdot \underbrace{\log(x)'}_{f'}\, dx =$$

$$= \log(x) \cdot x\Big|_a^b - \int_a^b 1\, dx = x(\log(x) - 1)\Big|_a^b$$

Integrating rational functions

$f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$, $Q(x)$ – polynomials with real coefficients

Example $\displaystyle\int_1^3 \frac{1}{x\,(x^2+1)}\,dx$

Want to write:

unknown coefficients

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} =$$

$$= \frac{A(x^2+1) + x(Bx+C)}{x(x^2+1)}$$

So to get an equality

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

We need to have:

$$0 \cdot x^2 + 0 \cdot x + 1 = Ax^2 + A + Bx^2 + Cx$$

$$= (A+B)x^2 + Cx + A$$

$$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow A=1, \ B=-1, \ C=0$$

So we get

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$\int_1^3 \frac{1}{x(x^2+1)}\,dx = \int_1^3 \frac{1}{x}\,dx - \int_1^3 \frac{x}{x^2+1}\,dx$$

$$= \log|x|\Big|_1^3 - \frac{1}{2}\log(x^2+1)\Big|_1^3 =$$

$$= \left(\log(x) - \frac{1}{2}\log(x^2+1) \right) \Bigg|_1^3 =$$

$$= \log(3) - \underbrace{\cancel{\log(1)}}_{0} - \left(\frac{1}{2}\log(10) - \frac{1}{2}\log(2) \right)$$

$$= \log 3 - \frac{1}{2}\log(10) + \frac{1}{2}\log(2) \qquad \blacksquare$$

In general,

Proposition 8.43 Any rational function $\frac{P(x)}{Q(x)}$ can be written as

$$\frac{P(x)}{Q(x)} = \alpha_1 R_1(x) + \dots + \alpha_t R_t(x)$$

with $\alpha_i \in \mathbb{R}$ and $R_i(x)$ of the form:

1) polynomial, 2) $\frac{1}{(x-r)^p}$, 3) $\frac{x+c}{(x^2+2rx+s)^p}$

How to find this decomposition?

1st step if $\deg P \geq \deg Q$

then we can produce division with remainder

to get $$\frac{P(x)}{Q(x)} = \underbrace{R_1(x)} + \frac{\tilde{P}(x)}{Q(x)}$$

$\deg \tilde{P} < \deg Q$

We can assume $\deg P < \deg Q$

then 2nd step:

find factorisation of Q

Fundamental theorem of algebra over $\mathbb{R}$:

$Q = \prod F_i$ where $F_i = (x-r)^p$

or $(x^2+2\alpha x+c)^p$

3rd step

Use undeterment coeffisients method to compute coefficients α_i.